

# Lattices and Periodic Geodesics in Pseudoriemannian 2-step Nilpotent Lie Groups

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7 Feb 2008

**Abstract:** We give a basic treatment of lattices  $\Gamma$  in these groups. Certain tori  $T_F$  and  $T_B$  provide the model fiber and the base for a submersion of  $\Gamma \backslash N$ . This submersion may not be pseudoriemannian in the usual sense, because the tori may be degenerate. We then begin the study of periodic geodesics in these compact nilmanifolds, obtaining a complete calculation of the period spectrum of certain flat spaces.

Appeared in *Int. J. Geom. Methods Mod. Phys.* **5** (2008) 79–99.

MSC(1991): Primary 53C50; Secondary 22E25, 53B30, 53C30.



<sup>1</sup>Partially supported by Project MEC:MTM2005-08757-C04-01, Spain.

<sup>2</sup>Partially supported by MEC:DGES Program SAB1995-0757, Spain.

# 1 Introduction

The 2-step nilpotent groups are nonabelian and as close as possible to being Abelian, but display a rich variety of new and interesting geometric phenomena. As in the Riemannian case, one of many places where they arise naturally is as groups of isometries acting on horospheres in certain (pseudoriemannian) symmetric spaces. Another is in the Iwasawa decomposition of semisimple groups with the Killing metric, which need not be definite. Here we study the lattices and periodic geodesics in these group spaces. For a more extensive historical introduction that better puts them into the contemporary context, see [7, 8]. A recent, masterful survey of the Riemannian case is [11].

By an *inner product* on a vector space  $V$  we shall mean a nondegenerate, symmetric bilinear form on  $V$ , generally denoted by  $\langle \cdot, \cdot \rangle$ . In particular, we *do not* assume that it is positive definite. Our convention is that  $v \in V$  is timelike if  $\langle v, v \rangle > 0$ , null if  $\langle v, v \rangle = 0$ , and spacelike if  $\langle v, v \rangle < 0$ .

Throughout,  $N$  will denote a connected, 2-step nilpotent Lie group with Lie algebra  $\mathfrak{n}$  having center  $\mathfrak{z}$ . (Recall that 2-step means  $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}$ .) We shall use  $\langle \cdot, \cdot \rangle$  to denote either an inner product on  $\mathfrak{n}$  or the induced left-invariant pseudoriemannian (indefinite) metric tensor on  $N$ .

In Section 2, we give the fundamental definitions and examples used in the rest of this paper. The main problem encountered is that the center  $\mathfrak{z}$  of  $\mathfrak{n}$  may be degenerate: it might contain a (totally) null subspace. We shall see that this possible degeneracy of the center causes the essential differences between the Riemannian and pseudoriemannian cases.

Much like the Riemannian case, we would expect that  $(N, \langle \cdot, \cdot \rangle)$  should in some sense be similar to flat pseudoeuclidean space. This is seen in the examples of totally geodesic subgroups in Section 3. We also showed [7] the existence of  $\dim \mathfrak{z}$  independent first integrals, a familiar result in pseudoeuclidean space. Unlike the Riemannian case, there are flat groups which are isometric to pseudoeuclidean spaces.

Section 4 begins with a basic treatment of lattices  $\Gamma$  in these groups. The tori  $T_F$  and  $T_B$  provide the model fiber and the base for a submersion of  $\Gamma \backslash N$ . This submersion may not be pseudoriemannian in the usual sense, because the tori may be degenerate. Also, it is possible for a (null) geodesic to be closed but not periodic. Thus we begin the study of periodic geodesics in such a compact nilmanifold and of its period spectrum. In the Riemannian case, this is its length spectrum. We obtain a complete calculation of the period spectrum for the flat spaces of Section 2. For related work on the length spectrum in the Riemannian setting (which is closely related to

the geometry of the Laplacian and which plays a central role in isospectral questions), we refer to [12, 13, 14, 18, 20, 21].

We recall [7, 11] some basic facts about 2-step nilpotent Lie groups. As with all nilpotent Lie groups, the exponential map  $\exp : \mathfrak{n} \rightarrow N$  is surjective. Indeed, it is a diffeomorphism for simply connected  $N$ ; in this case we shall denote the inverse by  $\log$ . The Baker-Campbell-Hausdorff formula takes on a particularly simple form in these groups:

$$\exp(x) \exp(y) = \exp(x + y + \tfrac{1}{2}[x, y]). \quad (1)$$

Letting  $L_n$  denote left translation by  $n \in N$ , we have the following:

**Lemma 1.1** *Let  $\mathfrak{n}$  denote a 2-step nilpotent Lie algebra and  $N$  the corresponding simply connected Lie group. If  $x, a \in \mathfrak{n}$ , then*

$$\exp_{x*}(a_x) = L_{\exp(x)*} \left( a + \tfrac{1}{2}[a, x] \right)$$

where  $a_x$  denotes the initial velocity vector of the curve  $t \mapsto x + ta$ .

**Corollary 1.2** *In a pseudoriemannian 2-step nilpotent Lie group, the exponential map preserves causal character. Alternatively, 1-parameter subgroups are curves of constant causal character.*

**Proof:** For the 1-parameter subgroup  $c(t) = \exp(ta)$ , one easily sees that  $\dot{c}(t) = \exp_{ta*}(a) = L_{\exp(ta)*}a$  and left translations are isometries.  $\square$

Of course, 1-parameter subgroups need not be geodesics, as simple examples show [7].

We shall also need some basic facts about lattices in  $N$ . In nilpotent Lie groups, a lattice is a discrete subgroup  $\Gamma$  such that the homogeneous space  $M = \Gamma \backslash N$  is compact [26]. Lattices do not always exist in nilpotent Lie groups [19].

**Theorem 1.3** *The simply connected, nilpotent Lie group  $N$  admits a lattice if and only if there exists a basis of its Lie algebra  $\mathfrak{n}$  for which the structure constants are rational.*

Such a group is said to have a rational structure, or simply to be rational.

Geodesic completeness is notoriously problematic in pseudoriemannian spaces. For 2-step nilpotent Lie groups, things work nicely as shown by this result first published by Guediri [15].

**Theorem 1.4** *On a 2-step nilpotent Lie group, all left-invariant pseudoriemannian metrics are geodesically complete.*

He also provided an explicit example of an incomplete metric on a 3-step nilpotent Lie group; we refer to [7] for complete calculations of explicit formulas for the connection, curvatures, covariant derivative, *etc.*

## 2 Definitions and Examples

For the convenience of the reader, we repeat some basic definitions and theorems from [8]. Complete details and proofs are in [7].

In the Riemannian (positive-definite) case, one splits  $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{z} \oplus \mathfrak{z}^\perp$  where the superscript denotes the orthogonal complement with respect to the inner product  $\langle, \rangle$ . In the general pseudoriemannian case, however,  $\mathfrak{z} \oplus \mathfrak{z}^\perp \neq \mathfrak{n}$ . The problem is that  $\mathfrak{z}$  might be a degenerate subspace; *i.e.*, it might contain a null subspace  $\mathfrak{U}$  for which  $\mathfrak{U} \subseteq \mathfrak{U}^\perp$ .

Thus we have to adopt a more complicated decomposition of  $\mathfrak{n}$ . Observe that if  $\mathfrak{z}$  is degenerate, the null subspace  $\mathfrak{U}$  is well defined invariantly. We use a decomposition

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$$

in which  $\mathfrak{z} = \mathfrak{U} \oplus \mathfrak{Z}$  and  $\mathfrak{v} = \mathfrak{V} \oplus \mathfrak{E}$ ,  $\mathfrak{U}$  and  $\mathfrak{V}$  are complementary null subspaces, and  $\mathfrak{U}^\perp \cap \mathfrak{V}^\perp = \mathfrak{Z} \oplus \mathfrak{E}$ . Although the choice of  $\mathfrak{V}$  is *not* well defined invariantly, once a  $\mathfrak{V}$  has been chosen then  $\mathfrak{Z}$  and  $\mathfrak{E}$  are well defined invariantly. Indeed,  $\mathfrak{Z}$  is the portion of the center  $\mathfrak{z}$  in  $\mathfrak{U}^\perp \cap \mathfrak{V}^\perp$  and  $\mathfrak{E}$  is its orthocomplement in  $\mathfrak{U}^\perp \cap \mathfrak{V}^\perp$ . This is a Witt decomposition of  $\mathfrak{n}$  given  $\mathfrak{U}$  as described in [25, p. 37f], easily seen by noting that  $(\mathfrak{U} \oplus \mathfrak{V})^\perp = \mathfrak{Z} \oplus \mathfrak{E}$ , adapted to the special role of the center in  $\mathfrak{n}$ .

We fix a choice of  $\mathfrak{V}$  (and therefore  $\mathfrak{Z}$  and  $\mathfrak{E}$ ). Having fixed  $\mathfrak{V}$ , observe that the inner product  $\langle, \rangle$  provides a dual pairing between  $\mathfrak{U}$  and  $\mathfrak{V}$ ; *i.e.*, isomorphisms  $\mathfrak{U}^* \cong \mathfrak{V}$  and  $\mathfrak{U} \cong \mathfrak{V}^*$ . Thus the choice of a basis  $\{u_i\}$  in  $\mathfrak{U}$  determines an isomorphism  $\mathfrak{U} \cong \mathfrak{V}$  (*via* the dual basis  $\{v_i\}$  in  $\mathfrak{V}$ ). In addition to the choice of  $\mathfrak{V}$ , we also fix a basis of  $\mathfrak{U}$ .

We also need an involution  $\iota$  that interchanges  $\mathfrak{U}$  and  $\mathfrak{V}$  by this isomorphism and which reduces to the identity on  $\mathfrak{Z} \oplus \mathfrak{E}$  in the Riemannian (positive-definite) case. The choice of such an involution is not significant [7]. In terms of chosen orthonormal bases  $\{z_\alpha\}$  of  $\mathfrak{Z}$  and  $\{e_a\}$  of  $\mathfrak{E}$ ,

$$\iota(u_i) = v_i, \quad \iota(v_i) = u_i, \quad \iota(z_\alpha) = \varepsilon_\alpha z_\alpha, \quad \iota(e_a) = \bar{\varepsilon}_a e_a,$$

where, as usual,

$$\langle u_i, v_i \rangle = 1, \quad \langle z_\alpha, z_\alpha \rangle = \varepsilon_\alpha, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.$$

Then  $\iota(\mathfrak{U}) = \mathfrak{V}$ ,  $\iota(\mathfrak{V}) = \mathfrak{U}$ ,  $\iota(\mathfrak{Z}) = \mathfrak{Z}$ ,  $\iota(\mathfrak{E}) = \mathfrak{E}$  and  $\iota^2 = I$ . It is obvious that  $\iota$  is selfadjoint with respect to the inner product,

$$\langle \iota x, y \rangle = \langle x, \iota y \rangle, \quad x, y \in \mathfrak{n}, \quad (2)$$

so  $\iota$  is an isometry of  $\mathfrak{n}$ . (However, it does *not* integrate to an isometry of  $N$ ; see Example 2.9.) Moreover,

$$\langle x, \iota x \rangle = 0 \text{ if and only if } x = 0, \quad x \in \mathfrak{n}. \quad (3)$$

Consider the adjoint with respect to  $\langle \cdot, \cdot \rangle$  of the adjoint representation of the Lie algebra  $\mathfrak{n}$  on itself, denoted by  $\text{ad}^\dagger$ . First note that for all  $a \in \mathfrak{Z}$ ,  $\text{ad}_a^\dagger \bullet = 0$ . Thus for all  $y \in \mathfrak{n}$ ,  $\text{ad}_\bullet^\dagger y$  maps  $\mathfrak{V} \oplus \mathfrak{E}$  to  $\mathfrak{U} \oplus \mathfrak{E}$ . Moreover, for all  $u \in \mathfrak{U}$  we have  $\text{ad}_\bullet^\dagger u = 0$  and for all  $e \in \mathfrak{E}$  also  $\text{ad}_\bullet^\dagger e = 0$ . Following [17, 9, 10], we define the operator  $j$ . Note the use of the involution  $\iota$  to obtain a good analogy to the Riemannian case.

**Definition 2.1** The linear mapping  $j : \mathfrak{U} \oplus \mathfrak{Z} \rightarrow \text{End}(\mathfrak{V} \oplus \mathfrak{E})$  is given by  $j(a)x = \iota \text{ad}_x^\dagger \iota a$ .

Let  $x, y \in \mathfrak{n}$ . Recall [4, 7] that homaloidal planes are those for which the numerator  $\langle R(x, y)y, x \rangle$  of the sectional curvature  $K(x, y)$  vanishes. This notion is useful for degenerate planes tangent to spaces that are not of constant curvature.

**Theorem 2.2** *All central planes are homaloidal:  $R(z, z')z'' = R(u, x)y = R(x, y)u = 0$  for all  $z, z', z'' \in \mathfrak{Z}$ ,  $u \in \mathfrak{U}$ , and  $x, y \in \mathfrak{n}$ . Thus the nondegenerate part of the center is flat:  $K(z, z') = 0$ .*

In view of this result, we extend the notion of flatness to possibly degenerate submanifolds.

**Definition 2.3** A submanifold of a pseudoriemannian manifold is *flat* if and only if every plane tangent to the submanifold is homaloidal.

**Corollary 2.4** *The center  $Z$  of  $N$  is flat.*

**Corollary 2.5** *The only  $N$  of constant curvature are flat.*

The degenerate part of the center can have a profound effect on the geometry of the whole group.

**Theorem 2.6** *If  $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$  and  $\mathfrak{E} = \{0\}$ , then  $N$  is flat.*

We continue with a formula for certain sectional curvatures.

**Theorem 2.7** *If  $e, e'$  are any orthonormal vectors in  $\mathfrak{E}$ , then*

$$K(e, e') = -\frac{3}{4}\bar{\varepsilon}\bar{\varepsilon}'\langle [e, e'], [e, e'] \rangle$$

with  $\bar{\varepsilon} = \langle e, e \rangle$  and  $\bar{\varepsilon}' = \langle e', e' \rangle$ .

Some sectional curvature numerators are also relevant.

**Proposition 2.8** *If  $z \in \mathfrak{Z}$ ,  $v \in \mathfrak{V}$ , and  $e \in \mathfrak{E}$ , then*

$$\begin{aligned} \langle R(z, v)v, z \rangle &= \frac{1}{4}\langle j(\iota z)v, j(\iota z)v \rangle, \\ \langle R(v, e)e, v \rangle &= -\frac{3}{4}\langle [v, e], [v, e] \rangle + \frac{1}{4}\langle j(\iota v)e, j(\iota v)e \rangle, \\ \langle R(v, v')v', v \rangle &= -\frac{3}{4}\langle [v, v'], [v, v'] \rangle + \frac{1}{2}\langle j(\iota v)v', j(\iota v')v \rangle \\ &\quad + \frac{1}{4}\left( \langle j(\iota v')v, j(\iota v')v \rangle + \langle j(\iota v)v', j(\iota v)v' \rangle \right) \\ &\quad - \langle j(\iota v)v, j(\iota v')v' \rangle. \end{aligned}$$

Here is the example we mentioned previously; details and other examples are in [7, 8].

**Example 2.9** For the simplest quaternionic Heisenberg algebra of dimension 7, we may take a basis  $\{u_1, u_2, z, v_1, v_2, e_1, e_2\}$  with structure equations

$$\begin{aligned} [e_1, e_2] &= z & [v_1, v_2] &= z \\ [e_1, v_1] &= u_1 & [e_2, v_1] &= u_2 \\ [e_1, v_2] &= u_2 & [e_2, v_2] &= -u_1 \end{aligned}$$

and nontrivial inner products

$$\langle u_i, v_j \rangle = \delta_{ij}, \quad \langle z, z \rangle = \varepsilon, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.$$

As usual, each  $\varepsilon$ -symbol is  $\pm 1$  independently (this is a combined null and orthonormal basis), so the signature is  $(++--\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2)$ .

Sectional curvatures for this group are

$$\begin{aligned} \langle R(v_1, v_2)v_2, v_1 \rangle &= -(\bar{\varepsilon}_1 + \frac{3}{4}\varepsilon), \\ \langle R(v, e)e, v \rangle &= \langle R(z, v)v, z \rangle = 0, \\ K(z, e_1) &= K(z, e_2) = \frac{1}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2, \\ K(e_1, e_2) &= -\frac{3}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2. \end{aligned}$$

Thus  $\iota$  cannot integrate to an isometry of  $N$  in general, as mentioned after equation (2). Isometries must preserve vanishing of sectional curvature, and an integral of  $\iota$  would interchange homaloidal and nonhomaloidal planes in this example.

### 3 Totally Geodesic Subgroups and Geodesics

We begin by noting that O'Neill [22, Ex. 9, p.125] has extended the definition of totally geodesic to degenerate submanifolds of pseudoriemannian manifolds. We shall use this extended version.

Recall from [22] that the extrinsic and intrinsic curvatures of totally geodesic submanifolds coincide. Thus there is an unambiguous notion of flatness for them. Note that a connected subgroup  $N'$  of  $N$  is a totally geodesic submanifold if and only if it is totally geodesic at the identity element of  $N$ , because left translations by elements of  $N'$  are isometries of  $N$  that leave  $N'$  invariant. A connected, totally geodesic submanifold need not be a connected, totally geodesic subgroup, but  $(N, \langle, \rangle)$  has many totally geodesic subgroups. Many of them are flat, illustrating the similarity to pseudoeuclidean spaces; cf. [10, (2.11)].

**Example 3.1** For any  $x \in \mathfrak{n}$  the 1-parameter subgroup  $\exp(tx)$  is a geodesic if and only if  $\text{ad}_x^\dagger x = 0$ . We find this if and only if  $x \in \mathfrak{z}$  or  $x \in \mathfrak{U} \oplus \mathfrak{E}$ . This is essentially the same as the Riemannian case, but with some additional geodesic 1-parameter subgroups coming from  $\mathfrak{U}$ .

**Example 3.2** Abelian subspaces of  $\mathfrak{V} \oplus \mathfrak{E}$  are Lie subalgebras of  $\mathfrak{n}$ , and give rise to complete, flat, totally geodesic abelian subgroups of  $N$ , just as in the Riemannian case [10]. The construction given in [10, (2.11), Ex. 2] is valid in general, and shows that if  $\dim \mathfrak{V} \oplus \mathfrak{E} \geq 1 + k + k \dim \mathfrak{z}$ , then every nonzero element of  $\mathfrak{V} \oplus \mathfrak{E}$  lies in an abelian subspace of dimension  $k + 1$ .

**Example 3.3** The center  $Z$  of  $N$  is a complete, flat, totally geodesic submanifold. Moreover, it determines a foliation of  $N$  by its left translates, so each leaf is flat and totally geodesic, as in the Riemannian case [10]. In the pseudoriemannian case, this foliation in turn is the orthogonal direct sum of two foliations determined by  $\mathfrak{U}$  and  $\mathfrak{z}$ , and the leaves of the  $\mathfrak{U}$ -foliation are also null. All these leaves are complete.

For the geodesic equation, let us consider (as suffices) a geodesic  $\gamma$  with  $\gamma(0) = 1 \in N$  and  $\dot{\gamma}(0) = a_0 + x_0 \in \mathfrak{z} \oplus \mathfrak{v}$ . One may further decompose  $a_0 = u_0 + z_0 \in \mathfrak{U} \oplus \mathfrak{z}$  and  $x_0 = v_0 + e_0 \in \mathfrak{V} \oplus \mathfrak{E}$ . In exponential coordinates, write  $\gamma(t) = \exp(a(t) + x(t)) = \exp(u(t) + z(t) + v(t) + e(t))$  where one has  $\dot{a}(0) = a_0$ ,  $\dot{x}(0) = x_0$ , etc. For the tangent vector  $\dot{\gamma}$ , we obtain

$$\begin{aligned} \dot{\gamma} &= \exp_{(a+x)*}(\dot{a} + \dot{x}) \\ &= L_{\gamma(t)*} \left( \dot{a} + \dot{x} + \frac{1}{2}[\dot{a} + \dot{x}, a + x] \right) \\ &= L_{\gamma(t)*} \left( \dot{a} + \dot{x} + \frac{1}{2}[\dot{x}, x] \right), \end{aligned}$$

using Lemma 1.1, regarded as vector fields along  $\gamma$ . Then the geodesic equation is equivalent to

$$\frac{d}{dt} \left( \dot{a} + \dot{x} + \frac{1}{2}[\dot{x}, x] \right) - \text{ad}_{\dot{a} + \dot{x} + \frac{1}{2}[\dot{x}, x]}^\dagger \left( \dot{a} + \dot{x} + \frac{1}{2}[\dot{x}, x] \right) = 0.$$

Simplifying slightly, we find

$$\underbrace{\frac{d}{dt} \left( \dot{a} + \frac{1}{2}[\dot{x}, x] \right)}_{\in \mathfrak{u} \oplus \mathfrak{z}} + \underbrace{\ddot{x}}_{\in \mathfrak{v} \oplus \mathfrak{e}} - \underbrace{\text{ad}_x^\dagger \left( \dot{z} + \dot{v} + \frac{1}{2}[\dot{x}, x] \right)}_{\in \mathfrak{u} \oplus \mathfrak{e}} = 0. \quad (4)$$

Using superscripts to denote components, we obtain for the  $\mathfrak{z}$ -component

$$\frac{d}{dt} \left( \dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{z}} \right) = 0.$$

Using the initial condition, we get

$$\dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{z}} = z_0.$$

Next we note that  $\ddot{v} = 0$  whence  $\dot{v}(t) = v_0$  is a constant. We use these to simplify the other component equations.

$$\frac{d}{dt} \left( \dot{u} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{u}} \right) - \left( \text{ad}_x^\dagger (z_0 + v_0) \right)^{\mathfrak{u}} = 0 \quad (5)$$

$$\dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{z}} = z_0 \quad (6)$$

$$\ddot{v} = 0 \quad (7)$$

$$\ddot{e} - \left( \text{ad}_x^\dagger (z_0 + v_0) \right)^{\mathfrak{e}} = 0 \quad (8)$$

In analogy with Eberlein [9, 10] we define two operators.

**Definition 3.4** For fixed  $z_0 \in \mathfrak{z}$  and  $v_0 \in \mathfrak{v}$  as above, define

$$\mathcal{J} : \mathfrak{v} \oplus \mathfrak{e} \longrightarrow \mathfrak{e} : y \longmapsto \left( \text{ad}_y^\dagger (z_0 + v_0) \right)^{\mathfrak{e}},$$

$$\mathcal{J} : \mathfrak{v} \oplus \mathfrak{e} \longrightarrow \mathfrak{u} : y \longmapsto \left( \text{ad}_y^\dagger (z_0 + v_0) \right)^{\mathfrak{u}}.$$

We shall denote the restriction of  $\mathcal{J}$  to  $\mathfrak{e}$  by  $J$ , and this will play the same role as  $J$  in Eberlein [9, 10].



Now we rewrite the geodesic equations in terms of  $J$  and  $\mathcal{J}$ , using the linearity of  $\text{ad}^\dagger$  to rearrange some terms.

$$\frac{d}{dt}(\dot{u} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{U}}) = \mathcal{J}\dot{x} \quad (9)$$

$$\dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{Z}} = z_0 \quad (10)$$

$$\ddot{v} = 0 \quad (11)$$

$$\ddot{e} - J\dot{e} = \mathcal{J}v_0 \quad (12)$$

While the  $\mathfrak{V}$ -component of a geodesic is simple, its mere presence affects all of the other components.

We also readily see that the system is completely integrable. Thus, as noted in the Introduction (Theorem 1.4), all left-invariant pseudoriemannian metrics on these groups are complete. Also, *regardless of signature*, we may obtain the existence of  $\dim \mathfrak{z}$  first integrals as in [10].

Keep  $J \in \text{End}(\mathfrak{E})$  and observe that  $J$  is skewadjoint with respect to  $\langle \cdot, \cdot \rangle$ . Write  $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$  with  $\mathfrak{E}_1 = \ker J$  as a direct sum. Unfortunately, it need not be orthogonal; see [16] for a complete list of canonical forms for such  $J$ . Thus we shall assume for the rest of this article that  $\mathfrak{E}_1 \oplus \mathfrak{E}_2$  is orthogonal. For example, this will be the case if  $\ker J$  is nondegenerate.

Decompose  $\mathcal{J}v_0 = y_1 + y_2 \in \mathfrak{E}_1 \oplus \mathfrak{E}_2$ , respectively. Note that  $J$  is invertible on  $\mathfrak{E}_2$ ; we denote this restriction by  $J$  also.

We continue with a geodesic through the identity element, so  $\gamma(0) = 1$  and  $\dot{\gamma}(0) = a_0 + x_0 = u_0 + z_0 + v_0 + e_0$ , with  $J \in \text{End}(\mathfrak{E})$  and  $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$  with  $\mathfrak{E}_1 = \ker J$ , and with  $\mathcal{J}v_0 = y_1 + y_2 \in \mathfrak{E}_1 \oplus \mathfrak{E}_2$  as before. Decompose  $e_0 = e_1 + e_2 \in \mathfrak{E}_1 \oplus \mathfrak{E}_2$ , respectively. (These  $e_i$  should not be confused with the basis elements appearing in other sections.) Recall that  $J$  is invertible on  $\mathfrak{E}_2$ , and that we let  $J$  denote the restriction there as well. For convenience, set  $x_1 = e_1 + v_0 - J^{-1}y_2$  and  $x_2 = e_2 + J^{-1}y_2$ .

**Theorem 3.5** *Under these assumptions, the geodesic equations may be integrated as:*

$$x(t) = tx_1 + (e^{tJ} - I)J^{-1}x_2 + \frac{1}{2}t^2y_1, \quad (13)$$

$$z(t) = tz_0 + \mathcal{J}[\dot{x}, x]^{\mathfrak{Z}}, \quad (14)$$

$$u(t) = tu_0 + \mathcal{J}[\dot{x}, x]^{\mathfrak{U}} + \mathcal{J}\mathcal{J}\dot{x}, \quad (15)$$

where

$$\begin{aligned}
\mathcal{J}[\dot{x}, x] &= -\frac{1}{2} \int_0^t [\dot{x}(s), x(s)] ds \\
&= \frac{1}{2}t \left[ x_1 + \frac{1}{2}ty_1, (e^{tJ} + I) J^{-1}x_2 \right] - t \left[ y_1, e^{tJ}x_2 \right] + \frac{1}{12}t^3[x_1, y_1] \\
&\quad + \frac{1}{2} \left[ (e^{tJ} - I) J^{-1}x_2, J^{-1}x_2 \right] - \left[ x_1, (e^{tJ} - I) J^{-2}x_2 \right] \\
&\quad + \left[ y_1, (e^{tJ} - I) J^{-3}x_2 \right] + \frac{1}{2} \int_0^t \left[ e^{sJ} J^{-1}x_2, e^{sJ}x_2 \right] ds
\end{aligned}$$

and

$$\mathcal{L}\mathcal{J}\dot{x} = \int_0^t \int_0^s \mathcal{J}\dot{x}(\sigma) d\sigma ds.$$

**Proof:** The formulas follow from straightforward integrations of the geodesic equations (9)–(12). We used the general fact about exponentials of matrices that  $J$  commutes with  $e^{tJ}$  for all  $t \in \mathbb{R}$ . Using this, it is routine to verify that  $x(t)$ ,  $z(t)$ , and  $u(t)$  satisfy the geodesic equations and initial conditions.  $\square$

**Corollary 3.6** *When  $N$  has a nondegenerate center, the formulas simplify somewhat. Now equation (12) is homogeneous and we obtain*

$$e(t) = te_1 + (e^{tJ} - I) J^{-1}e_2, \quad (16)$$

$$z(t) = tz_1(t) + z_2(t) + z_3(t), \quad (17)$$

where

$$\begin{aligned}
z_1(t) &= z_0 + \frac{1}{2} \left[ e_1, (e^{tJ} + I) J^{-1}e_2 \right], \\
z_2(t) &= \left[ e_1, (I - e^{tJ}) J^{-2}e_2 \right] + \frac{1}{2} \left[ e^{tJ} J^{-1}e_2, J^{-1}e_2 \right], \\
z_3(t) &= \frac{1}{2} \int_0^t \left[ e^{sJ} J^{-1}e_2, e^{sJ}e_2 \right] ds.
\end{aligned}$$

Note that  $z_3$  may contribute to  $z_1$  and  $z_2$ .  $\square$

The flat spaces we found in Theorem 2.6 allow more simplification.

**Corollary 3.7** *When  $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$  and  $\mathfrak{E} = \{0\}$ , then  $x = v$  and we obtain*

$$v(t) = tv_0, \quad (18)$$

$$z(t) = tz_0, \quad (19)$$

$$u(t) = tu_0 + \frac{1}{2}t^2 \mathcal{J}v_0. \quad (20)$$

**Corollary 3.8** *If  $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$  and  $\mathfrak{E} = \{0\}$ , then  $N$  is geodesically connected. Consequently, so is any nilmanifold with such a universal covering space.*

**Proof:**  $N$  is complete by Theorem 1.4, hence pseudoconvex. The preceding geodesic equations show that  $N$  is nonreturning. Thus the space of geodesics  $G(N)$  is Hausdorff by Theorem 5.2 of [5]. Now Theorem 4.2 of [3] yields geodesic connectedness of  $N$ .  $\square$

Thus these compact nilmanifolds are much like tori. This is also illustrated by the computation of their period spectrum in Theorem 4.25.

## 4 Lattices and Periodic Geodesics

In this section, we assume that  $N$  is rational and let  $\Gamma$  be a lattice in  $N$ . Since  $N$  is 2-step nilpotent, it has nice generating sets of  $\Gamma$ ; cf. [9, (5.3)].

**Proposition 4.1** *If  $N$  is a simply connected, 2-step nilpotent Lie group of dimension  $n$  with lattice  $\Gamma$  and with center  $Z$  of dimension  $m$ , then there exists a canonical generating set  $\{\varphi_1, \dots, \varphi_n\}$  such that  $\{\varphi_1, \dots, \varphi_m\}$  generate  $\Gamma \cap Z$ . In particular,  $\Gamma \cap Z$  is a lattice in  $Z$ .*

From this, formula (1), and [26, Prop. 2.17], one obtains as in [10, (5.3)]

**Corollary 4.2** *Let  $N$  be a simply connected, 2-step nilpotent Lie group with lattice  $\Gamma$  and let  $\pi : \mathfrak{n} \rightarrow \mathfrak{v}$  denote the projection. Then*

1.  $\log \Gamma \cap \mathfrak{z}$  is a vector lattice in  $\mathfrak{z}$ ;
2.  $\pi(\log \Gamma)$  is a vector lattice in  $\mathfrak{v}$ ;
3.  $\Gamma \cap Z = Z(\Gamma)$ .  $\square$

Here, we used the splitting  $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$  from Section 2 with  $\mathfrak{z} = \mathfrak{U} \oplus \mathfrak{Z}$  and  $\mathfrak{v} = \mathfrak{V} \oplus \mathfrak{E}$ . Thus to the compact nilmanifold  $\Gamma \backslash N$  we may associate two flat (possibly degenerate) tori; cf. [10, p.644].

**Definition 4.3** With notation as preceding,

$$\begin{aligned} T_{\mathfrak{z}} &= \mathfrak{z} / (\log \Gamma \cap \mathfrak{z}), \\ T_{\mathfrak{v}} &= \mathfrak{v} / \pi(\log \Gamma). \end{aligned}$$

Observe that  $\dim T_{\mathfrak{z}} + \dim T_{\mathfrak{v}} = \dim \mathfrak{z} + \dim \mathfrak{v} = \dim \mathfrak{n}$ .

Next, we apply Theorem 3 from [23] to our situation. Let  $T^m$  denote the  $m$ -torus as usual.

**Theorem 4.4** *Let  $m = \dim \mathfrak{z}$  and  $n = \dim \mathfrak{v}$ . Then  $\Gamma \backslash N$  is a principal  $T^m$ -bundle over  $T^n$ .*

The model fiber  $T^m$  can be given a geometric structure from its closed embedding in  $\Gamma \backslash N$ ; we denote this geometric  $m$ -torus by  $T_F$ . Similarly, we wish to provide the base  $n$ -torus with a geometric structure so that the projection  $p_B : \Gamma \backslash N \rightarrow T_B$  is the appropriate generalization of a pseudoriemannian submersion [22] to (possibly) degenerate spaces. Observe that the splitting  $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$  induces splittings  $TN = \mathfrak{z}N \oplus \mathfrak{v}N$  and  $T(\Gamma \backslash N) = \mathfrak{z}(\Gamma \backslash N) \oplus \mathfrak{v}(\Gamma \backslash N)$ , and that  $p_{B*}$  just mods out  $\mathfrak{z}(\Gamma \backslash N)$ . Examining the definition on page 212 of [22], we see that the key is to construct the geometry of  $T_B$  by defining

$$p_{B*} : \mathfrak{v}_\eta(\Gamma \backslash N) \rightarrow T_{p_B(\eta)}(T_B) \text{ for each } \eta \in \Gamma \backslash N \text{ is an isometry} \quad (21)$$

and

$$\nabla_{p_{B*}x}^{T_B} p_{B*}y = p_{B*}(\pi \nabla_x y) \text{ for all } x, y \in \mathfrak{v} = \mathfrak{V} \oplus \mathfrak{E}, \quad (22)$$

where  $\pi : \mathfrak{n} \rightarrow \mathfrak{v}$  is the projection. Then the rest of the results of pages 212–213 in [22] will continue to hold, provided that sectional curvature is replaced by the numerator of the sectional curvature formula in his 47. Theorem at least when elements of  $\mathfrak{V}$  are involved:

$$\langle R_{T_B}(p_{B*}x, p_{B*}y)p_{B*}y, p_{B*}x \rangle = \langle R_{\Gamma \backslash N}(x, y)y, x \rangle + \frac{3}{4}\langle [x, y], [x, y] \rangle. \quad (23)$$

Now  $p_B$  will be a pseudoriemannian submersion in the usual sense if and only if  $\mathfrak{U} = \mathfrak{V} = \{0\}$ , as is always the case for Riemannian spaces.

In the Riemannian case, Eberlein showed that  $T_F \cong T_{\mathfrak{z}}$  and  $T_B \cong T_{\mathfrak{v}}$ . Observe that it follows from Theorem 2.7, Proposition 2.8, and (23) that  $T_B$  is flat in general only if  $N$  has a nondegenerate center or is flat.

**Proposition 4.5** *If  $v, v' \in \mathfrak{V}$  and  $e, e' \in \mathfrak{E}$ , then*

$$\begin{aligned} \langle R_{T_B}(v, e)e, v \rangle &= \frac{1}{4}\langle j(\iota v)e, j(\iota v)e \rangle, \\ \langle R_{T_B}(v, v')v', v \rangle &= \frac{1}{2}\langle j(\iota v)v', j(\iota v')v \rangle - \langle j(\iota v)v, j(\iota v')v' \rangle \\ &\quad + \frac{1}{4}\left(\langle j(\iota v')v, j(\iota v')v \rangle + \langle j(\iota v)v', j(\iota v)v' \rangle\right), \\ K_{T_B}(e, e') &= 0. \end{aligned}$$

*Here we have suppressed  $p_{B*}$  on the left-hand sides for simplicity.*

Our quaternionic Heisenberg group (Example 2.9) provides a simple example of a  $T_B$  that is not flat. Indeed, all sectional curvature numerators vanish except  $\langle R_{T_B}(v_1, v_2)v_2, v_1 \rangle = -\bar{\varepsilon}_1$ , where we have again suppressed  $p_{B*}$  for simplicity.

**Remark 4.6** Observe that the torus  $T_B$  may be decomposed into a topological product  $T_E \times T_V$  in the obvious way. It is easy to check that  $T_E$  is flat and isometric to  $(\log \Gamma \cap \mathfrak{E}) \backslash \mathfrak{E}$ , and that  $T_V$  has a linear connection not coming from a metric and not flat in general. Moreover, the geometry of the product is “twisted” in a certain way. It would be interesting to determine which tori could appear as such a  $T_V$  and how.

We wish to show that the geometry of the fibers  $T_F$  is that of  $T_3$ . Thus we now consider the submersion  $\Gamma \backslash N \twoheadrightarrow T_B$  and realize the model fiber  $T^m$  as  $(\Gamma \cap Z) \backslash Z$  considered as tori only, without geometry.

**Definition 4.7** Let  $p_N : N \rightarrow \Gamma \backslash N$  and  $p_Z : Z \rightarrow T^m = (\Gamma \cap Z) \backslash Z$  be the natural projections. Define  $F : T^m \rightarrow I(\Gamma \backslash N)$  by

$$F(p_Z(z))(p_N(n)) = p_N(zn) \quad \forall \quad z \in Z, n \in N.$$

We recall from [8] that  $I^{spl}(N)$  denotes the subgroup of the isometry group  $I(N)$  which preserves the splitting  $TN = \mathfrak{z}N \oplus \mathfrak{v}N$ .

**Proposition 4.8**  *$F$  is a smooth isomorphism of groups  $T^m \cong I_0^{spl}(\Gamma \backslash N)$ , where the subscript 0 denotes the identity component.*

**Proof:** We follow Eberlein [10, (5.4)]. It is easy to check that  $F$  is a well-defined, smooth, injective homomorphism with image in  $I_0^{spl}(\Gamma \backslash N)$ . Thus we need only show that  $F$  is surjective.

Let  $\psi \in I_0^{spl}(\Gamma \backslash N)$  and let  $\phi_t$  be a path from  $1 = \phi_0$  to  $\psi = \phi_1$ . The covering map  $p_N$  has the homotopy lifting property, so choose a lifting  $\tilde{\phi}_t$  as a path in  $I_0^{spl}(N)$ . Then for all  $g \in \Gamma$ , it follows that  $p_N(\tilde{\phi}_t L_g \tilde{\phi}_t^{-1}) = p_N$  for all  $t$ . Hence for each  $t \in [0, 1]$ , there exists  $g_t \in \Gamma$  such that  $\tilde{\phi}_t L_g \tilde{\phi}_t^{-1} = L_{g_t}$ . Since  $g_0 = g$  and  $\Gamma$  is a discrete group, it follows that  $g_t = g$  for every  $t$ , so  $L_g$  commutes with  $\tilde{\phi}_t$  for every  $g \in \Gamma$  and  $t \in [0, 1]$ .

From Proposition 3.3 in [8], there exist  $n_t \in N$  and  $a_t \in O(N)$  such that  $\tilde{\phi}_t = L_{n_t} a_t$  for all  $t$ . Now, every  $L_g$  commutes with every  $\tilde{\phi}_t$ , such that  $a_t(g) = n_t^{-1} g n_t$  for all  $t$  and  $g$ . Extension from lattices is unique [26, Thm. 2.11], so  $a_t = \text{Ad}_{n_t^{-1}}$ . By Lemma 3.1 in [8],  $a_t$  is the identity and  $n_t \in Z$  for all  $t$ . Thus  $\tilde{\phi}_1 = L_{n_1}$ , so from the definition of  $\tilde{\phi}_t$  we obtain  $p_N L_{n_1} = p_N \tilde{\phi}_1 = \phi_1 p_N = \psi p_N$ . But this means  $F(p_Z(n_1)) = \psi$ .  $\square$

**Corollary 4.9**  *$I_0^{spl}(\Gamma \backslash N)$  acts freely on  $\Gamma \backslash N$  with complete, flat, totally geodesic orbits.*

**Proof:** By Theorem 4.4, we may identify  $I_0^{spl}(\Gamma \backslash N)$  as the group of the principal bundle  $\Gamma \backslash N \twoheadrightarrow T_B$ , so it acts freely on the total space.

Since  $p_N$  is a local isometry and the  $Z$ -orbits in  $N$  are complete, flat, and totally geodesic from Example 3.3, it follows (using the identification  $T^m = (\Gamma \cap Z) \backslash Z$  *supra*) that the  $I_0^{spl}(\Gamma \backslash N)$ -orbits are complete, flat, and totally geodesic.  $\square$

**Theorem 4.10** *Let  $N$  be a simply connected, 2-step nilpotent Lie group with lattice  $\Gamma$ , a left-invariant metric tensor, and tori as in the discussion following Theorem 4.4. The fibers  $T_F$  of the (generalized) pseudoriemannian submersion  $\Gamma \backslash N \twoheadrightarrow T_B$  are isometric to  $T_{\mathfrak{z}}$ .*

**Proof:** We follow the proof of Eberlein [10, (5.5), item 2]. For each  $n \in N$ , define  $\psi_n = p_N L_n \exp : \mathfrak{z} \rightarrow \Gamma \backslash N$ , and note that it is a local isometry. Clearly,  $\psi_n(z) = \psi_n(z')$  if and only if  $z' = z + \log g$  for some  $g \in \Gamma \cap Z$ . Hence  $\psi_n$  induces an isometric embedding  $\tilde{\psi}_n : T_{\mathfrak{z}} \rightarrow \Gamma \backslash N$ . That the image is the  $I_0^{spl}(\Gamma \backslash N)$ -orbit of  $p_N(n)$  follows from the proof of Corollary 4.9.  $\square$

**Corollary 4.11** *If in addition the center  $Z$  of  $N$  is nondegenerate, then  $T_B$  is isometric to  $T_{\mathfrak{v}}$ .*  $\square$

The proof is essentially the same as the appropriate parts of the proof of (5.5) in [10] and we omit it.

We recall that elements of  $N$  can be identified with elements of the isometry group  $I(N)$ : namely,  $n \in N$  is identified with the isometry  $\phi = L_n$  of left translation by  $n$ . We shall abbreviate this by writing  $\phi \in N$ .

**Definition 4.12** We say that  $\phi \in N$  *translates* the geodesic  $\gamma$  by  $\omega$  if and only if  $\phi\gamma(t) = \gamma(t + \omega)$  for all  $t$ . If  $\gamma$  is a unit-speed geodesic, we say that  $\omega$  is a *period* of  $\phi$ .

Recall that unit speed means that  $|\dot{\gamma}| = |\langle \dot{\gamma}, \dot{\gamma} \rangle|^{\frac{1}{2}} = 1$ . Since there is no natural normalization for null geodesics, we do not define periods for them. In the Riemannian case and in the timelike Lorentzian case in strongly causal spacetimes [2], unit-speed geodesics are parameterized by arclength and this period is a translation distance. If  $\phi$  belongs to a lattice  $\Gamma$ , it is the length of a closed geodesic in  $\Gamma \backslash N$ .

**Remark 4.13** Note that it follows from Corollary 1.2 that if  $\phi = \exp(a^* + x^*)$  translates a geodesic  $\gamma$  with  $\gamma(0) = 1 \in N$ , then  $a^* + x^*$  and  $\dot{\gamma}(0)$  are of the same causal character.

In general, recall that if  $\gamma$  is a geodesic in  $N$  and if  $p_N : N \twoheadrightarrow \Gamma \backslash N$  denotes the natural projection, then  $p_N \gamma$  is a periodic geodesic in  $\Gamma \backslash N$  if and only if some  $\phi \in \Gamma$  translates  $\gamma$ . We say *periodic* rather than *closed* here because in pseudoriemannian spaces it is possible for a null geodesic to be closed but not periodic. If the space is geodesically complete or Riemannian, however, then this does not occur (*cf.* [22], p. 193); the former is in fact the case for our 2-step nilpotent Lie groups by Theorem 1.4. Further recall that free homotopy classes of closed curves in  $\Gamma \backslash N$  correspond bijectively with conjugacy classes in  $\Gamma$ .

**Definition 4.14** Let  $\mathcal{C}$  denote either a nontrivial, free homotopy class of closed curves in  $\Gamma \backslash N$  or the corresponding conjugacy class in  $\Gamma$ . We define  $\wp(\mathcal{C})$  to be the set of all periods of periodic unit-speed geodesics that belong to  $\mathcal{C}$ .

In the Riemannian case, this is the set of lengths of closed geodesics in  $\mathcal{C}$ , frequently denoted by  $\ell(\mathcal{C})$ .

**Definition 4.15** The *period spectrum* of  $\Gamma \backslash N$  is the set

$$\text{spec}_{\wp}(\Gamma \backslash N) = \bigcup_{\mathcal{C}} \wp(\mathcal{C}),$$

where the union is taken over all nontrivial, free homotopy classes of closed curves in  $\Gamma \backslash N$ . Note that In the Riemannian case, this is the length spectrum  $\text{spec}_{\ell}(\Gamma \backslash N)$ .

**Example 4.16** Similar to the Riemannian case, we can compute the period spectrum of a flat torus  $\Gamma \backslash \mathbb{R}^m$ , where  $\Gamma$  is a lattice (of maximal rank, isomorphic to  $\mathbb{Z}^m$ ). Using calculations related to those of [6, pp. 146–8] in an analogous way as for finding the length spectrum of a Riemannian flat torus, we easily obtain

$$\text{spec}_{\wp}(\Gamma \backslash \mathbb{R}^m) = \{|g| \neq 0 \mid g \in \Gamma\}.$$

It is also easy to see that the nonzero d'Alembertian spectrum is related to the analogous set produced from the dual lattice  $\Gamma^*$  as multiples by  $\pm 4\pi^2$ , almost as in the Riemannian case.

As in this example, simple determinacy of periods of unit-speed geodesics helps make calculation of the period spectrum possible purely in terms of  $\log \Gamma \subseteq \mathfrak{n}$ . (See Theorem 4.25 for another example.) Thus we begin with the following observation.

**Proposition 4.17** *Let  $\phi = \exp(a^* + v^* + e^*)$  translate the unit-speed geodesic  $\gamma$  by  $\omega > 0$ . If  $v^* \neq 0$ , then the period  $\omega$  is simply determined.*

**Proof:** We may assume  $\gamma(0) = 1 \in N$  and  $\dot{\gamma}(0) = a_0 + v_0 + e_0$ . From Theorem 3.5,  $v^* = v(\omega) = \omega v_0$ .  $\square$

In attempting to calculate the period spectrum then, we can focus our attention on those cases where the  $\mathfrak{V}$ -component is zero.

From now on, we assume that  $N$  is a simply connected, 2-step nilpotent Lie group with left-invariant pseudoriemannian metric tensor  $\langle \cdot, \cdot \rangle$ . Note that non-null geodesics may be taken to be of unit speed. Most nonidentity elements of  $N$  translate some geodesic, but not necessarily one of unit speed; cf. [10, (4.2)].

**Proposition 4.18** *Let  $N$  be a simply connected, 2-step nilpotent Lie group with left-invariant metric tensor  $\langle \cdot, \cdot \rangle$  and  $\phi \in N$  not the identity. Write  $\log \phi = a^* + x^* \in \mathfrak{z} \oplus \mathfrak{v}$  and assume that  $x^* \perp [x^*, \mathfrak{n}]$ . Let  $a'$  be the component of  $a^*$  orthogonal to  $[x^*, \mathfrak{n}]$  in  $\mathfrak{z}$  and choose  $\xi \in \mathfrak{n}$  such that  $a' = a^* + [x^*, \xi]$ . Set  $\omega^* = |a' + x^*|$  if  $a' + x^*$  is not null and set  $\omega^* = 1$  otherwise. Then  $\phi$  translates the geodesic*

$$\gamma(t) = \exp(\xi) \exp\left(\frac{t}{\omega^*}(a' + x^*)\right)$$

*by  $\omega^*$ , and  $\gamma$  is of unit speed if  $a' + x^*$  is not null.*

**Proof:** Let  $n = \exp(\xi)$ . One may set  $\phi^* = n^{-1}\phi n = \exp(a' + x^*)$  and one may set  $\gamma^*(t) = n^{-1}\gamma(t)$ . Then  $\phi\gamma(t) = \gamma(t + \omega^*)$  is equivalent to  $\phi^*\gamma^*(t) = \gamma^*(t + \omega^*)$ , and the latter is routine to verify using (1). Now  $\gamma$  is a geodesic if and only if  $\gamma^*$  is, and it is easy to check directly that  $\nabla_{a' + x^*}(a' + x^*) = 0$  is equivalent to  $\langle a' + x^*, [x^*, \mathfrak{n}] \rangle = 0$ .  $\square$

Note that the  $\mathfrak{U}$  components of  $a^*$  and  $a'$  in fact coincide. Also note that if we further decompose  $x^* = v^* + e^*$ , the result applies to every nonidentity element  $\phi$  with  $v^* = 0$ . In particular, when the center is nondegenerate this is every nonidentity element.

**Corollary 4.19** *When  $\mathfrak{n}$  is nonsingular and  $\phi \notin Z$ , we may take  $a' = 0$  in Proposition 4.18.*

**Proof:** Because then  $a^* \in [x^*, \mathfrak{n}]$  and  $x^* \neq 0$ .  $\square$



Now we give some general criteria for an element  $\phi$  to translate a geodesic  $\gamma$ ; cf. [10, (4.3)]. We use a  $J$  as in the passage following Definition 3.4, and  $x_1, x_2, y_1$ , and  $y_2$  as given just before Theorem 3.5.

**Proposition 4.20** *Let  $\phi \in N$  and write  $\phi = \exp(a^* + x^*)$  for suitable elements  $a^* \in \mathfrak{z}$  and  $x^* \in \mathfrak{v}$ . Let  $\gamma$  be a geodesic with  $\gamma(0) = n \in N$  and  $\gamma(\omega) = \phi n$ , and let  $\dot{\gamma}(0) = L_{n*}(a_0 + x_0)$  for suitable elements  $a_0 \in \mathfrak{z}$  and  $x_0 = v_0 + e_0 \in \mathfrak{v}$ . Let  $n^{-1}\gamma(t) = \exp(a(t) + x(t))$  where  $a(t) \in \mathfrak{z}$  and  $x(t) \in \mathfrak{v}$  for all  $t \in \mathbb{R}$  and  $a(0) = x(0) = 0$ . Then the following are equivalent:*

1.  $\left. \begin{aligned} x(t + \omega) &= x(t) + x^* \\ a(t + \omega) &= a(t) + a^* + \frac{1}{2}[x^*, x(t)] \end{aligned} \right\} \text{ for all } t \in \mathbb{R} \text{ and some } \omega > 0;$
2.  $\gamma(t + \omega) = \phi\gamma(t)$  for all  $t \in \mathbb{R}$  and some  $\omega > 0$ ;
3.  $e^{\omega J}$  fixes  $e_1 + y_1 + x_2 = e_0 + y_1 + J^{-1}y_2$ .

**Proof:** As before, we may assume without loss of generality  $n = 1 \in N$ . Items 1 and 2 are equivalent by formula (1). The following lemma shows that item 1 implies item 3.

**Lemma 4.21** *As in the preamble to Theorem 3.5, assume  $\mathfrak{E}$  as an orthogonal direct sum  $\mathfrak{E}_1 \oplus \mathfrak{E}_2$  with  $\mathfrak{E}_1 = \ker J$ , and use  $x_1, x_2, y_1$ , and  $y_2$  as given there. Then  $x^* = \omega x_1 + \frac{1}{2}\omega^2 y_1$  and  $e^{\omega J}$  fixes  $x_2$ .*

**Proof:** By Theorem 3.5,  $x(k\omega) = k\omega x_1 + (e^{k\omega J} - I)J^{-1}x_2 + \frac{1}{2}k^2\omega^2 y_1$  for every positive integer  $k$ . By induction, from item 1 in the statement of the proposition we obtain  $x(k\omega) = kx^*$  for every  $k$ . One may decompose  $x^* = v^* + e_1^* + e_2^* \in \mathfrak{V} \oplus \mathfrak{E}_1 \oplus \mathfrak{E}_2$  to see  $v^* = \omega v_0$ ,  $e_1^* = \omega e_1 + \frac{1}{2}\omega^2 y_1$ , and

$$k(e_2^* + \omega J^{-1}y_2) = (e^{k\omega J} - I)J^{-1}x_2 \quad \forall k. \quad (24)$$

for every  $k$ .

Now,  $e^{\omega J}$  is an element of the identity component of the pseudorthogonal group of isometries of  $\langle, \rangle$ , and as such can be decomposed into a product of reflections, ordinary rotations, and boosts. With respect to appropriate coordinates, which may be different from our standard choice, a boost will have a matrix of the form

$$\begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix}$$

on some pair of basis vectors, for some  $s \in \mathbb{R}$ .

If  $e^{\omega J}$  is composed only of reflections and ordinary rotations, then the right-hand side of (24) is uniformly bounded in  $k$  (say, with respect to the positive definite  $\langle \cdot, \cdot \rangle$ ) while the left-hand side is unbounded, we see that  $e_2^* + \omega J^{-1}y_2 = 0$ . On the other hand, if  $e^{\omega J}$  is a pure boost, then the right-hand side grows exponentially in  $k$  while the left-hand side grows but linearly, and again we obtain  $e_2^* + \omega J^{-1}y_2 = 0$ . The Lemma now follows from this and (24) for  $k = 1$ .  $\square$

The proof that item 3 implies item 2 is the same as the relevant part of the proof of (4.3) in [10].  $\square$

**Corollary 4.22** *When in addition  $v^* = v_0 = 0$ , the following are also equivalent to the three items in Proposition 4.20.*

1.  $\dot{\gamma}(0)$  is orthogonal to the orbit  $Z_{e^*}n$ , where  $Z_{e^*} = \exp([e^*, \mathfrak{n}]) \subseteq Z$ ;
2.  $\dot{\gamma}(\omega)$  is orthogonal to the orbit  $Z_{e^*}\phi n$ .

**Proof:** Note that under this hypothesis,  $x^* = e^*$ . Now Lemma 4.21 implies that  $J(e^*) = 0$ , and this is now equivalent to  $z_0 \perp [e^*, \mathfrak{n}]$ . Thus the relevant parts of the proof of (4.3) in [10] apply *mutatis mutandis*.  $\square$

We also obtain the following results as in Eberlein [10, (4.4), (4.9)]. Note that we assume that  $v^* = v_0 = 0$  in the first, but that this is automatic in the second.

**Corollary 4.23** *Let  $\phi \in N$  and write  $\phi = \exp(a^* + e^*)$  for unique elements  $a^* \in \mathfrak{z}$  and  $e^* \in \mathfrak{E}$ . Let  $n \in N$  be given and write  $n = \exp(\xi)$  for a unique  $\xi \in \mathfrak{n}$ . Then the following are equivalent:*

1. *There exists a geodesic  $\gamma$  in  $N$  with  $\gamma(0) = n$  such that  $\phi\gamma(t) = \gamma(t+\omega)$  for all  $t \in \mathbb{R}$  and some  $\omega > 0$ .*
2. *There exists a geodesic  $\gamma^*$  in  $N$  with  $\gamma^*(0) = 1$ ,  $\dot{\gamma}^*(0)$  is orthogonal to  $[e^*, \mathfrak{n}]$ , and  $\gamma^*(\omega) = \exp([e^*, \xi])\phi$  for some  $\omega > 0$ .*  $\square$

**Corollary 4.24** *Let  $1 \neq \phi \in Z$  and  $\gamma$  be any geodesic so  $\gamma(\omega) = \phi\gamma(0)$  for some  $\omega > 0$ . Then  $\phi\gamma(t) = \gamma(t+\omega)$  for all  $t \in \mathbb{R}$ .*  $\square$

In the flat 2-step nilmanifolds of Theorem 2.6, we can calculate the period spectrum completely.

**Theorem 4.25** *If  $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$  and  $\mathfrak{E} = \{0\}$ , then  $\text{spec}_\varphi(M)$  can be completely calculated from  $\log \Gamma$  for any  $M = \Gamma \backslash N$ .*

**Proof:** Let  $\phi$  translate a unit-speed geodesic  $\gamma$  by  $\omega > 0$ . As usual, we may as well assume that  $\gamma(0) = 1 \in N$ . Write  $\log \phi = a^* + v^*$  and  $\dot{\gamma}(0) = a_0 + v_0$ . From Corollary 3.7 we have that  $v^* = v(\omega) = \omega v_0$ ,  $z^* = z(\omega) = \omega z_0$ , and  $u^* = u(\omega) = \omega u_0 + \frac{1}{2}\omega^2 \mathcal{J}v_0$ . Note that

$$\omega^2 \mathcal{J}v_0 = \omega^2 \operatorname{ad}_{v_0}^\dagger (z_0 + v_0) = \omega^2 \operatorname{ad}_{v_0}^\dagger v_0 = \operatorname{ad}_{v^*}^\dagger v^*.$$

Substituting and rearranging, we obtain  $\varepsilon\omega^2 = 2\langle u^*, v^* \rangle + \langle z^*, z^* \rangle$ , where  $\pm 1 = \varepsilon = \langle \dot{\gamma}(0), \dot{\gamma}(0) \rangle = 2\langle u_0, v_0 \rangle + \langle z_0, z_0 \rangle$ .  $\square$

Thus we see again, as mentioned after Corollary 3.7, just how much these flat, 2-step nilmanifolds are like tori. All periods can be calculated purely from  $\log \Gamma \subseteq \mathfrak{n}$ , although some will not show up from the tori in the fibration.

**Corollary 4.26**  $\operatorname{spec}_\varphi(T_B)$  (respectively,  $T_F$ ) is  $\cup_{\mathcal{C}} \varphi(\mathbb{C})$  where the union is taken over all those free homotopy classes  $\mathcal{C}$  of closed curves in  $M = \Gamma \backslash N$  that do not (respectively, do) contain an element in the center of  $\Gamma \cong \pi_1(M)$ , except for those periods arising only from unit-speed geodesics in  $M$  that project to null geodesics in both  $T_B$  and  $T_F$ .  $\square$

We note that one might consider using this to assign periods to some null geodesics in the tori  $T_B$  and  $T_F$ .

When the center is nondegenerate, we obtain results similar to Eberlein's [10, (4.5)].

**Proposition 4.27** Assume  $\mathfrak{U} = \{0\}$ . Let  $\phi \in N$  and write  $\log \phi = z^* + e^*$ . Assume  $\phi$  translates the unit-speed geodesic  $\gamma$  by  $\omega > 0$ . Let  $z'$  denote the component of  $z^*$  orthogonal to  $[e^*, \mathfrak{n}]$ . Let  $n = \gamma(0)$  and set  $\omega^* = |z' + e^*|$ . Let  $\dot{\gamma}(0) = L_{n^*}(z_0 + e_0)$  and use  $J$ ,  $e_1$ , and  $e_2$  as in Corollary 3.6 (see also just before Theorem 3.5). Then

1.  $|e^*| \leq \omega$ . In addition,  $\omega < \omega^*$  for timelike (spacelike) geodesics with  $\omega z_0 - z'$  timelike (spacelike), and  $\omega > \omega^*$  for timelike (spacelike) geodesics with  $\omega z_0 - z'$  spacelike (timelike).
2.  $\omega = |e^*|$  if and only if  $\gamma(t) = \exp(t e^*/|e^*|)$  for all  $t \in \mathbb{R}$ .
3.  $\omega = \omega^*$  if and only if  $\omega z_0 - z'$  is null. If moreover  $\omega^* z_0 = z'$ , then  $e_2 = 0$  if and only if
  - (a)  $\gamma(t) = n \exp(t(z' + e^*)/\omega^*)$  for all  $t \in \mathbb{R}$ .
  - (b)  $z' = z^* + [e^*, \xi]$  where  $\xi = \log n$ .

Although  $\omega^*$  need not be an upper bound for periods as in the Riemannian case, it nonetheless plays a special role among all periods, as seen in item 3 above, and we shall refer to it as the *distinguished* period associated with  $\phi \in N$ . When the center is definite, for example, we do have  $\omega \leq \omega^*$ .

**Proof:** As usual, we may assume that  $\gamma(0) = 1 \in N$ . Note that this replaces  $\phi$  as given in the statement with  $n^{-1}\phi n$  and  $\gamma$  with  $n^{-1}\gamma$ .

For the first part of item 1, since  $|\dot{\gamma}(0)| = 1$  there exists an orthonormal basis of  $\mathfrak{n}$  having  $\dot{\gamma}(0)$  as a member. (This may well be a different basis from our usual one.) Fix one such basis, and consider the positive-definite inner product with matrix  $I$  on this basis. Let  $\|\cdot\|$  denote the norm associated to this positive-definite inner product. By Lemma 4.21,  $e^* = \omega e_1$ . Then  $|e_1| \leq \|e_1\| \leq \|\dot{\gamma}(0)\| = 1$  so  $|e^*| = \omega|e_1| \leq \omega$ .

For the rest of item 1, we begin with Corollary 3.6 and get

$$\begin{aligned} e(t) &= t e_1 + (e^{tJ} - I) J^{-2} e_2, \\ z(t) &= t \left( z_0 + \frac{1}{2} [e_1, (e^{tJ} + I) J^{-1} e_2] \right) + z_2(t) + z_3(t). \end{aligned}$$

By Lemma 4.21,  $e^* = \omega e_1$  and  $e^{\omega J} e_2 = e_2$ . Inspecting the formula for  $z_2(t)$  in Corollary 3.6, we find  $z_2(\omega) = 0$ . Thus

$$\begin{aligned} z^* &= z(\omega) = \omega \left( z_0 + [e_1, J^{-1} e_2] \right) + z_3(\omega) \\ &= \omega z_0 + [e^*, J^{-1} e_2] + \frac{1}{2} \int_0^\omega [e^{sJ} J^{-1} e_2, e^{sJ} e_2] ds. \end{aligned}$$

By item 1 of Corollary 4.22,  $z_0 \perp [e^*, \mathfrak{n}]$ . Then

$$\langle z', z_0 \rangle = \langle z^*, z_0 \rangle = \omega \langle z_0, z_0 \rangle + \frac{1}{2} \int_0^\omega \langle [e^{sJ} J^{-1} e_2, e^{sJ} e_2], z_0 \rangle ds.$$

Recall that  $J$  is skewadjoint with respect to  $\langle \cdot, \cdot \rangle$  (whence  $e^{tJ}$  is an isometry of  $\langle \cdot, \cdot \rangle$  for all  $t$ ), that  $J$  commutes with every  $e^{tJ}$  (whence so does  $J^{-1}$ ), and that  $Jx = \text{ad}_x^\dagger z_0$ . We compute

$$\begin{aligned} \langle [e^{sJ} J^{-1} e_2, e^{sJ} e_2], z_0 \rangle &= - \langle [e^{sJ} e_2, e^{sJ} J^{-1} e_2], z_0 \rangle \\ &= - \langle J^{-1} e^{sJ} e_2, J e^{sJ} e_2 \rangle \\ &= \langle e^{sJ} e_2, e^{sJ} e_2 \rangle \\ &= \langle e_2, e_2 \rangle. \end{aligned}$$

Therefore,

$$\langle z', z_0 \rangle = \omega \langle z_0, z_0 \rangle + \frac{\omega}{2} \langle e_2, e_2 \rangle. \quad (25)$$

Now  $|\dot{\gamma}(0)| = 1$  so  $\varepsilon = \langle z_0, z_0 \rangle + \langle e_1, e_1 \rangle + \langle e_2, e_2 \rangle$ , where  $\varepsilon = \pm 1$  as usual. Substituting in (25) for  $\langle e_2, e_2 \rangle$ , we obtain

$$\langle z', z_0 \rangle = \frac{\omega}{2}(\varepsilon + \langle z_0, z_0 \rangle) - \frac{\omega}{2} \frac{\langle e^*, e^* \rangle}{\omega^2}$$

so  $\langle e^*, e^* \rangle - \varepsilon \omega^2 = \omega^2 \langle z_0, z_0 \rangle - 2\omega \langle z', z_0 \rangle$ . Adding  $\langle z', z' \rangle$  to both sides,

$$\langle z' + e^*, z' + e^* \rangle - \varepsilon \omega^2 = \langle \omega z_0 - z', \omega z_0 - z' \rangle. \quad (26)$$

There are several cases:  $\varepsilon$  is 1 or  $-1$  and  $\omega z_0 - z'$  is timelike, spacelike, or null. If  $\omega z_0 - z'$  is null, then  $|\langle z' + e^*, z' + e^* \rangle| = \omega^2 > 0$  and  $\omega = |z' + e^*|$ . If  $\varepsilon = 1$  and  $\omega z_0 - z'$  is timelike, or if  $\varepsilon = -1$  and  $\omega z_0 - z'$  is spacelike, then  $\varepsilon \langle z' + e^*, z' + e^* \rangle > \omega^2 > 0$  whence  $\omega < |z' + e^*|$ . If  $\varepsilon = 1$  and  $\omega z_0 - z'$  is spacelike, or if  $\varepsilon = -1$  and  $\omega z_0 - z'$  is timelike, then it follows similarly that  $\omega > |z' + e^*|$ . This completes the proof of item 1.

Now we prove item 2. If  $\gamma$  is as given there, then  $\omega = |e^*|$  because  $\exp(z^* + e^*) = \phi = \gamma(\omega) = \exp(\omega e^*/|e^*|)$ . Conversely, assume  $\omega = |e^*|$  and consider the associated positive-definite inner product  $\langle \cdot, \iota \cdot \rangle$ . Changing the basis of  $\mathfrak{E}$  if necessary, we may assume that  $\mathfrak{Z}$ ,  $\mathfrak{E}_1$ , and  $\mathfrak{E}_2$  are mutually orthogonal with respect to both  $\langle \cdot, \cdot \rangle$  and  $\langle \cdot, \iota \cdot \rangle$ . Let  $\|\cdot\|$  now denote the norm for  $\langle \cdot, \iota \cdot \rangle$ . Then

$$\|\dot{\gamma}(0)\|^2 = \|z_0\|^2 + \|e_1\|^2 + \|e_2\|^2 \quad (27)$$

so  $\|\dot{\gamma}(0)\|^2 = \|e_1\|^2$  if and only if  $\dot{\gamma}(0) = e_1 = e^*/|e^*|$ . But now  $\gamma$  has the same initial data as  $\exp(t e^*/|e^*|)$ , so by uniqueness they must coincide.

Finally, we prove the last part of item 3; the first part is immediate from the last part of the proof of item 1 above. So assume  $\omega^* z_0 - z' = 0$  or  $z_0 = z'/\omega^*$ . Continue with the immediately previous positive-definite norm  $\|\cdot\|$  and basis of  $\mathfrak{E}$ . Substituting in (27) we get

$$\|\dot{\gamma}(0)\|^2 = \frac{\|z'\|^2}{(\omega^*)^2} + \frac{\|e^*\|^2}{(\omega^*)^2} + \|e_2\|^2 = \frac{\|z' + e^*\|^2}{(\omega^*)^2} + \|e_2\|^2$$

whence  $\dot{\gamma}(0) = \frac{z' + e^*}{\omega^*}$  if and only if  $e_2 = 0$ .  $\square$

**Corollary 4.28** *Assume the center is nondegenerate. Let  $\phi \in N$  with  $\phi \notin Z$  and suppose that  $z^* \in [e^*, \mathfrak{n}]$ . Then*

1. *If  $\phi$  translates a timelike (spacelike) geodesic with  $z_0$  nonspacelike (nontimelike), then  $\phi$  has the unique period  $|e^*|$ .*

2. Let  $\gamma$  be a unit-speed geodesic in  $N$  with  $\gamma(0) = n = \exp(\xi)$  for a unique  $\xi \in \mathfrak{n}$ . Then  $\phi$  translates  $\gamma$  by the unique period  $|e^*| > 0$  if and only if  $[\xi, e^*] = z^*$  and  $\gamma(t) = n \exp(t e^* / |e^*|)$  for all  $t \in \mathbb{R}$ .

In particular, this applies to all noncentral  $\phi \in N$  if  $\mathfrak{n}$  is nonsingular.  $\square$

The proof follows that of [10, (4.6)] *mutatis mutandis* and we omit the details. From item 2 of Proposition 4.27, using Lemma 1.1, we obtain

$$\begin{aligned} \exp(e^* + \tfrac{1}{2}[\xi, e^*]) &= n \exp(e^*) = \gamma(|e^*|) = \phi n \\ &= \exp(z^* + e^*) \exp(\xi) = \exp(z^* + e^* + \tfrac{1}{2}[e^*, \xi]), \end{aligned}$$

thus avoiding the use of item 3 here. Anent the last comment, note that if  $\mathfrak{n}$  is nonsingular then in fact  $z_0 = 0$  in item 1, because  $z_0 \perp [e^*, \mathfrak{n}] = \mathfrak{z}$ .

In view of the comment following Proposition 4.27 and Corollary 4.28, the following definitions make sense at least for  $N$  with a nondegenerate center.

**Definition 4.29** Let  $\mathcal{C}$  denote either a nontrivial, free homotopy class of closed curves in  $\Gamma \backslash N$  or the corresponding conjugacy class in  $\Gamma$ . We define  $\wp^*(\mathcal{C})$  to be the distinguished periods of periodic unit-speed geodesics that belong to  $\mathcal{C}$ .

**Definition 4.30** The *distinguished period spectrum* of  $\Gamma \backslash N$  is the set

$$\mathcal{D}\text{spec}_{\wp}(\Gamma \backslash N) = \bigcup_{\mathcal{C}} \wp^*(\mathcal{C}),$$

where the union is taken over all nontrivial, free homotopy classes of closed curves in  $\Gamma \backslash N$ .

Then as an immediate consequence of the preceding corollary, we get:

**Corollary 4.31** Assume the center is nondegenerate. If  $\mathfrak{n}$  is nonsingular, then  $\text{spec}_{\wp}(T_B)$  (respectively,  $T_F$ ) is precisely the period spectrum (respectively, the distinguished period spectrum) of those free homotopy classes  $\mathcal{C}$  of closed curves in  $M = \Gamma \backslash N$  that do not (respectively, do) contain an element in the center of  $\Gamma \cong \pi_1(M)$ , except for those periods arising only from unit-speed geodesics in  $M$  that project to null geodesics in both  $T_B$  and  $T_F$ .  $\square$

## Acknowledgments

Once again, Parker wishes to thank the Departamento at Santiago for its fine hospitality. He also thanks WSU for a Summer Research Fellowship during which part of this work was done, and for a Sabbatical Leave during which it was continued.

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